



Tilting the Primordial Power Spectrum with Bulk Viscosity

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Within the context of the cold dark matter model, current observations suggest that inflationary models which generate a tilted primordial power spectrum with negligible gravitational waves provide the most promising mechanism for explaining large scale clustering. The general form of the inflationary potential which produces such a spectrum is a hyperbolic function and is interpreted physically as a bulk viscous stress contribution to the energy-momentum of a perfect baryotropic fluid. This is equivalent to expanding the equation of state as a truncated Taylor series. Particle physics models which lead to such a potential are discussed.

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1 Introduction

The origin and evolution of large scale structure is one of the most important problems in cosmology today. It is widely accepted that the growth of small fluctuations by gravitational instability leads to structure formation. The inflationary paradigm, whilst providing possible solutions to a number of other problems associated with the hot big bang model, also produces a Gaussian, adiabatic fluctuation spectrum which is nearly, though not exactly, scale-invariant (Guth 1981; Albrecht & Steinhardt 1982; Linde 1982; Olive 1990; Liddle & Lyth 1993). Based on this prediction, the standard Cold Dark Matter (CDM) model of galaxy formation employs the flat, Harrison-Zel'dovich spectrum as an input parameter (Efstathiou 1990). The CDM model successfully accounts for small ($\leq 10h^{-1}$ Mpc) and intermediate ($10h^{-1}$ Mpc - $100h^{-1}$ Mpc) scale observations, if one introduces a bias in the distribution of luminous to dark matter (Davies *et al* 1985).¹

However, standard CDM has come under severe pressure from a number of recent observations (for a detailed review see Liddle & Lyth 1993). In particular, the APM angular galaxy-galaxy correlation function (Maddox *et al* 1991) and the IRAS QDOT redshift survey (Efstathiou *et al* 1991) indicate that there exists more large scale structure than that predicted by CDM. One possible resolution to this problem is to consider *tilted* CDM models. Here the primordial power spectrum is assumed to be of the form $P(k) \propto A_S^2(k)k \propto k^n$, where k is the comoving wavenumber of the Fourier expansion of the perturbation, A_S is the amplitude of the quantum fluctuation when it crosses the Hubble radius during the matter- or radiation-dominated eras and n is the power spectrum. Other possibilities involve the addition of a cosmological constant or a hot dark matter component (Liddle & Lyth 1993).

Inflation also produces a spectrum of gravitational wave (tensor) perturbations, whose amplitude may or may not be comparable to that of the scalar fluctuations. In this paper we shall concentrate on models which lead to tilted power spectra with a negligible gravitational wave component. There exists a wide range of observational constraints on the tilt arising from large angle ($\theta \geq 3^\circ$) microwave background anisotropies (Smoot *et al* 1992), galaxy clustering (Maddox *et al* 1990; Efstathiou *et al* 1990), peculiar velocity flows (Bertschinger & Dekel 1989; Dekel, Bertschinger & Faber 1990; Bertschinger *et al* 1990), high redshift quasars (Efstathiou & Rees 1988) and the red shift of structure formation. When combined together these observations strongly limit the allowed value of n . It has been shown that tilted CDM cannot fit all of the data simultaneously (Adams *et al* 1993; Liddle & Lyth 1993). For inflationary models in which gravitational wave production is negligible, a lower limit of $n > 0.7$ is partially consistent with the COBE 2-sigma upper limit and the bulk flow data, if the clustering and pairwise velocity data are ignored. In models where the gravitational wave contribution to the microwave background anisotropy is important, however, this limit is strengthened to $n > 0.84$. This is clearly inconsistent with the APM

¹The current value of the expansion rate is $H_0 = 100h$ km s⁻¹ Mpc⁻¹, where $0.4 \leq h \leq 1$.

However, analytical solutions have only been found for a limited number of specific examples, such as the special case $p/\rho = \text{constant}$ (Barrow 1990). For an arbitrary equation of state which satisfies the dominant energy condition ($\rho - p \geq 0$), it proves convenient to redefine the sum and difference of ρ and p in terms of the new functions

$$\dot{\phi}^2 \equiv \rho + p \quad \Longleftrightarrow \quad \phi(t) = \int^t dt' \sqrt{\rho(t') + p(t')} \quad (4)$$

$$2V \equiv \rho - p. \quad (5)$$

We may then rewrite Eqs. (1) and (2) as

$$H^2 = \frac{1}{3} \left(\frac{1}{2} \dot{\phi}^2 + V \right) - \frac{k}{a^2} \quad (6)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \quad (7)$$

which are the Einstein field equations for a minimally coupled scalar field, ϕ . Specifying the equation of state now becomes a question of choosing an appropriate functional form for the potential $V(\phi)$, and vice-versa. For example, an exponential potential is equivalent to $p/\rho = \text{constant}$ when $k = 0$.

Recent advances in the treatment of Eqs. (6) and (7) have been made by viewing the scalar field as the effective dynamical variable of the system (Muslimov 1990; Salopek & Bond 1990; Salopek & Bond 1991; Lidsey 1991; Lidsey 1993). From the definition $\rho \equiv \dot{\phi}^2/2 + V$, the time dependence can be eliminated by rewriting the scalar field equation (7) as

$$\rho' = -3H\dot{\phi}, \quad \dot{\phi} \neq 0, \quad (8)$$

where a prime denotes differentiation with respect to ϕ . This is consistent if ϕ does not oscillate (i.e. $\dot{\phi}$ does not pass through zero). It follows that $6H^2 = -\rho'X'/X$, where $X(\phi) \equiv a^2(\phi)$, and the Friedmann equation becomes

$$\rho'X' + 2\rho X = 6k. \quad (9)$$

The potential can be found immediately from the expression

$$V(\phi) = \rho(\phi) - \frac{1}{18} \frac{(\rho')^2}{H^2(\phi)} \quad (10)$$

once the forms of $\rho(\phi)$ and $X(\phi)$ are known.

When $k = 0$, $3H^2 = \rho$, and these field equations take the particularly simple form

$$2H'a' = -Ha, \quad 2H' = -\dot{\phi}, \quad (11)$$

thereby allowing the general solution, $a(\phi)$, to be expressed in terms of quadratures with respect to ϕ (Salopek & Bond 1991; Lidsey 1993). The expression for the potential reduces to an Hamilton-Jacobi differential equation in $H(\phi)$ of the form

$$V(\phi) = 3\dot{H}^2(\phi) - 2(H')^2. \quad (12)$$

Scalar Spectrum	Gravitational Waves Important	Gravitational Waves Negligible
Small Tilt	ϵ large $2\epsilon \approx \eta$	ϵ small $ \eta $ small
Significant Tilt	ϵ large $ \eta $ large	ϵ small $ \eta $ large

Table 1 - The table of correspondences illustrating the connection between tilt and the magnitude of the energy and friction parameters. The description 'large' implies significantly larger than zero (but still less than unity) and 'small' implies the parameter is very close to zero.

Inflation proceeds in the region of parameter space for which $\epsilon < 1$ and the coasting solution, or Milne universe, corresponds to $\epsilon = 1$ (i.e. $2y = x$). It is interesting to note that the friction parameter does not directly determine whether inflation occurs. The 'slow-roll' approximation is valid when $\{\epsilon, |\eta|\} \ll 1$.

One may also write the (scale-dependent) spectral indices of the scalar and tensor fluctuations in terms of these two quantities. It is easy to show that

$$n - 1 \equiv \frac{d \ln[A_S^2(k)]}{d \ln k} = 2 \left(\frac{2\epsilon_* - \eta_*}{\epsilon_* - 1} \right) \quad (19)$$

$$n_G \equiv \frac{d \ln[A_G^2(k)]}{d \ln k} = \frac{2\epsilon_*}{\epsilon_* - 1}, \quad (20)$$

where * indicates that ϵ and η should be evaluated when a particular scale first crosses the Hubble radius. The flat Harrison-Zel'dovich spectrum is equivalent to $n = 1$.

It follows from the definitions of A_S and A_G that

$$\frac{A_G}{A_S} = \frac{\sqrt{2}}{m} \sqrt{\epsilon} \quad (21)$$

It is often stated that inflation leads to a Harrison-Zel'dovich scalar spectrum with a negligible gravitational wave contribution. However such a conclusion follows because the slow-roll approximation is assumed *a priori*. Eq. (21) implies that the gravity wave amplitude can be comparable to A_S if ϵ is sufficiently large. There exists a table of correspondences which summarizes the four possibilities (Barrow & Liddle 1993).

In the following section we shall employ this framework to derive the form of the inflaton potential.

and

$$\epsilon = \left(\frac{n-1}{n-3} \right) \left[\tanh \left(\sqrt{\frac{(n-1)(n-3)}{8}} \phi \right) \right]^2 \quad (30)$$

for the exponential and hyperbolic cases respectively. If cosmological scales crossed the Hubble radius when $|\phi| \ll 1$, the amplitude of tensor perturbations is exponentially suppressed in the latter example.

Since Eq. (24) is an exact solution, it is valid for all values of ϕ . In particular, for sufficiently small ϕ the Taylor expansion

$$H(\phi) = \lambda \left[1 - \left(\frac{1-n}{8} \right) \phi^2 + \mathcal{O}(\phi^4) \right] \approx \sqrt{\frac{V(\phi)}{3}} \quad (31)$$

will also lead to a constant spectral index. This implies that a power spectrum with $n = \text{constant} < 1$ will arise from any function of $H(\phi)$ which is identical to Eq. (31) in the small ϕ approximation. It is well known that deviations from scale invariance, without significant gravitational wave production, are possible whenever the potential resembles an inverted harmonic oscillator (Steinhardt & Turner 1984). The above calculation provides further insight in the sense that such a result follows because the inverted oscillator resembles the hyperbolic secant function to lowest order. Eq. (31) is very useful because it directly relates the effective imaginary mass of the field to the scalar spectral index.

These results are summarized pictorially in Figures (1a) and (1b), which are representations of the class of solutions (23) in the $x-y$ plane. (x and y are defined in Eq. (18)). In figure (1a) the coasting solution $y = x/2$ is shown as the line M and the strong energy condition is violated to the right of this line. The x -axis represents the de Sitter solution, $H = \text{constant}$, and the origin is Minkowski space, which itself may be viewed as de Sitter space with an infinite radius of curvature (Hawking & Ellis 1973). The dashed lines represent solutions of constant n when $C = 0$. In these models x is a measure of the energy density of the universe and decreases as time increases. The trajectories of these constant n universes are indicated by the arrows and they all asymptotically approach Minkowski space at $t \rightarrow +\infty$.²

Figure 1

Figure (1b) illustrates the trajectories for finite values of $C < 0$ and $n = 0.7$. This class of universe begins in a de Sitter phase at $t = -\infty$ and evolves towards the $C = 0$ asymptote at $t = +\infty$ in such a way that the scalar spectral index remains constant at all times. The magnitude of C determines the amplitude of the scalar quantum fluctuations but *not* the scale dependence.

Having found the form of the potential required, it is now necessary to consider the physics which may lead to such a model.

²In reality the shape of the potential must change at some point (x, y) to allow for an exit from inflation.

where the constant of integration is expressed in terms of a_0 and $\{\alpha, \gamma\} \neq 0$. When $\beta = 0$, $\rho \propto a^{-3\gamma}$ as expected. Unfortunately the Friedmann equation (9) cannot be solved analytically when $k = \pm 1$, but has the exact solution

$$1 - \frac{\beta}{\gamma} \rho^a(\phi) = \tanh^2(\omega\phi) \quad (36)$$

$$\omega \equiv \sqrt{\frac{3\gamma\alpha^2}{4}} \quad (37)$$

when $k = 0$. The dominant energy condition is violated when $\beta < 0$, so this case is not considered further. The expressions for $H(\phi)$, $a(\phi)$ and $V(\phi)$ follow from Eqs. (11) and (12) as

$$H(\phi) = \frac{1}{\sqrt{3}} \left(\frac{\gamma}{\beta} \right)^{1/2\alpha} [\text{sech}(\omega\phi)]^{1/\alpha} \quad (38)$$

$$a(\phi) = a_0 \left(\frac{\beta}{\gamma} \right)^{1/3\alpha\gamma} |\sinh(\omega\phi)|^{2/3\alpha\gamma} \quad (39)$$

$$V(\phi) = \left(\frac{\gamma}{\beta} \right)^{1/\alpha} [\text{sech}(\omega\phi)]^{2/\alpha} \left(1 - (\gamma/2)\tanh^2(\omega\phi) \right). \quad (40)$$

These may be verified by differentiation. It should be emphasized that these solutions are exact and no ‘slow-roll’ approximations, such as $|\ddot{\phi}| \ll H|\dot{\phi}|$ and $\dot{\phi}^2 \ll V$, have been made. These parametric solutions are plotted schematically in Figure (2).

Figure 2

For completeness, we include the well known solutions for $\beta = 0$, which lead to the exponential potential

$$H(\phi) \propto \exp \left(\sqrt{\frac{3\gamma}{4}} \phi \right), \quad V(\phi) \propto \exp \left(\sqrt{3\gamma} \phi \right) \quad (41)$$

and power law expansion $a(t) \propto t^{2/3\gamma}$.

Hence, a bulk viscosity contribution to the baryotropic equation of state can be modelled as a self-interacting scalar field with potential (40) when $k = 0$. The inclusion of bulk viscosity alters the structure of the potential away from an exponential form in the neighbourhood of $|\phi| \approx 0$. This is illustrated in Fig. (2a). Near the origin, we find that the equation of state (34) can be adequately described as an inverted harmonic oscillator. The exponential potentials are recovered in the asymptotic limit as $|\phi| \rightarrow \infty$, because the viscous effects decay faster than the perfect fluid contribution at large $|\phi|$. The general advantage of rewriting Eq. (3) in terms of a scalar field is that the qualitative history of the universe is easily determined by considering the evolution of the scalar field along its interaction potential. Here, the field is initially placed at $\phi = 0$, which corresponds to a de Sitter expansion with $H = (\gamma/\beta)^{1/2\alpha}/\sqrt{3}$.

β of Eq. (34) plays the role of $|C|$ in determining the amplitude of the fluctuations, whereas α determines the tilt of the spectrum.

When $\{\alpha, \gamma\}$ are not related by Eq. (46) the tilt is not exactly scale invariant, but $n \approx \text{constant}$ is an excellent approximation if $\omega|\phi| \ll 1$. For standard reheating, scales of astrophysical interest first crossed the Hubble radius approximately 50 e-foldings before the end of inflation. Defining the value of the field at this point as ϕ_{50} , we find from Eq. (43) that

$$\sinh^2(\omega\phi_{50}) = \left(\frac{2}{3\gamma - 2} \right) e^{-3\alpha\gamma N_{50}} \quad (47)$$

for $\gamma > 2/3$, where $N_{50} \approx 50$. Hence, if γ is not too close to $\gamma = 2/3$, $\omega\phi_{50} \ll 1$ is valid and it is consistent to expand Eq. (38) to lowest order. It follows from Eq. (31) that

$$n = 1 - 3\alpha\gamma \quad (48)$$

and $n \geq 0.7$ leads to the constraint

$$\alpha\gamma \leq 0.1 \quad (49)$$

B. Quantum creation of fundamental strings

Turok (1988) has considered the quantum production of infinitely thin Witten strings on super-horizon size scales (Green, Schwarz & Witten 1988). He suggests that a deflationary expansion follows naturally from a quasi de Sitter phase in the early universe. It was further shown that the quantum creation of these strings after compactification to four dimensions is equivalent to the equation of state (34) when $\alpha = 1$ (Turok 1988). The parameter β depends on the fundamental string tension and the fractal dimension of the string. Moreover, a string distribution may be modelled in terms of a perfect fluid (32), where $2/3 \leq \gamma \leq 1$. The limits correspond to long strings with negligible velocity and a highly convoluted, relativistic string distribution respectively.

Therefore, the parametric solutions (38)-(40) describe the evolution of the flat Friedmann universe when dominated by fundamental strings created on super-horizon scales. As emphasized by Barrow (1988) this model is fragile in the sense that the de Sitter phase is replaced by an initial singularity if other matter components or anisotropies are included, and one might therefore view this model as unnatural. However, it is clear from constraint (49) that $\gamma < 0.1$ is required for consistency with observation if $\alpha = 1$ and this further undermines the attractiveness of this scenario. The model gives far too much power on large scales. On the other hand, if extra physics allows α to be significantly smaller ($\alpha \leq 0.15$), the model is not necessarily ruled out.

C. $N = 2$ supergravity in six dimensions

does provide a good fit to the data, then this would justify a more detailed study of the models discussed in section 5.

A number of simplifying assumptions were made. In particular, the expressions (13) and (14) for the scalar and tensor amplitudes are strictly only valid in the slow roll regime, $\{\epsilon, |\eta|\} \ll 1$, whereas Eq. (19) implies that a tilt away from the Harrison-Zel'dovich spectrum requires $0 \ll |\eta| \leq 1$, approximately. However, it has been shown that the corrections away from slow-roll are not important near a local maximum, and although they slightly alter the amplitude of the fluctuations in the exponential regime, they do not change the spectral index (Stewart & Lyth 1993).

The physical interpretation of the potential in terms of a bulk viscosity is only valid in the spatially flat FRW cosmology. However, this paper has investigated the power spectra of such models and it is the last 60 e-foldings of inflationary expansion which are important for large-scale structure (Kolb & Turner 1990). In most chaotic scenarios the density parameter is very close to unity by this stage.

Furthermore, since all scales probed by large-scale structure correspond to a small (≈ 9) number of e-foldings, it is reasonable to assume that the parameter γ is constant during this interval. These results may therefore have further applications in models where the polytropic index is a function of cosmic time.

Finally we note that Eq. (34) with $\gamma = 1$ and $\beta < 0$ is the equation of state for a polytropic star, special cases of which include white dwarfs ($\alpha = 5/3$) and neutron stars ($\alpha = 4/3$) (Weinberg 1973). If the solution to Eq. (9) could be found for $k = +1$, the techniques described here could be relevant for stellar structure.

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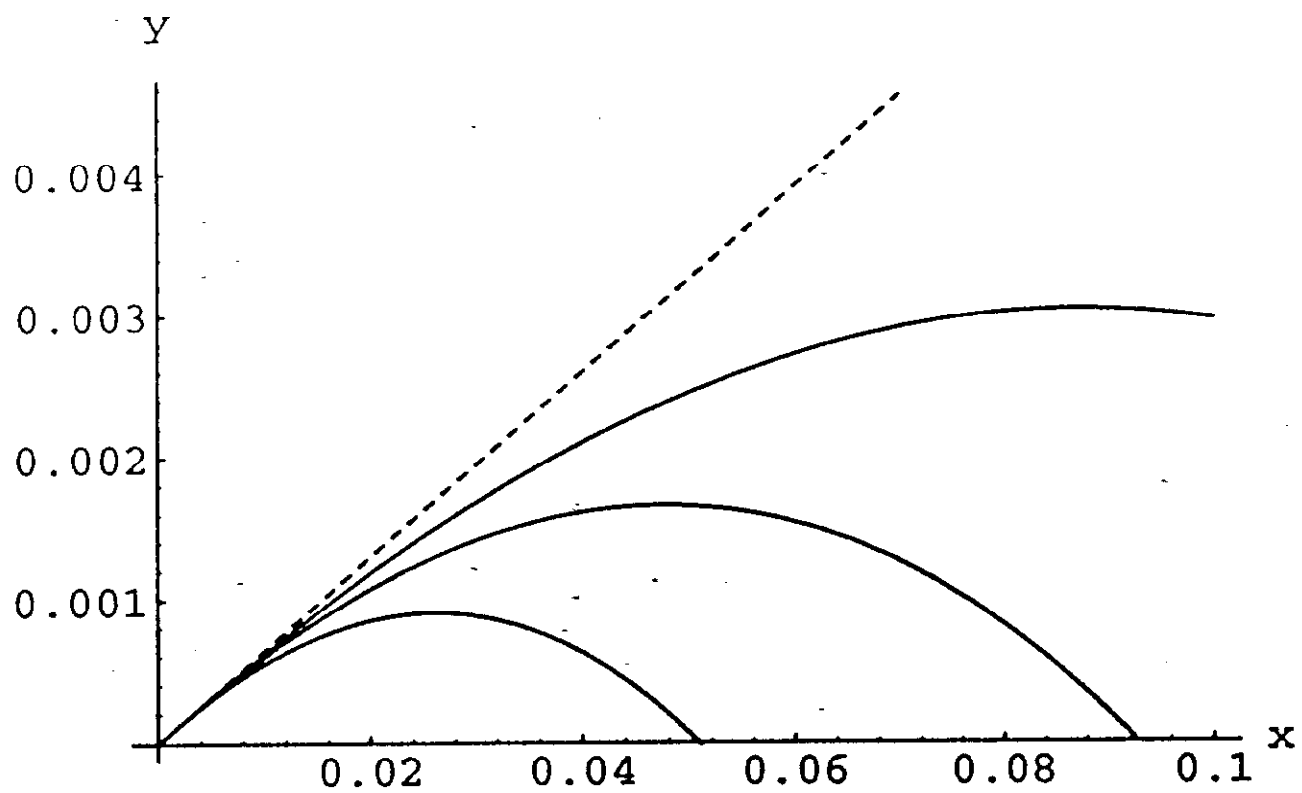


Figure 2.15